Optimal Spline Generators for Derivative Sampling

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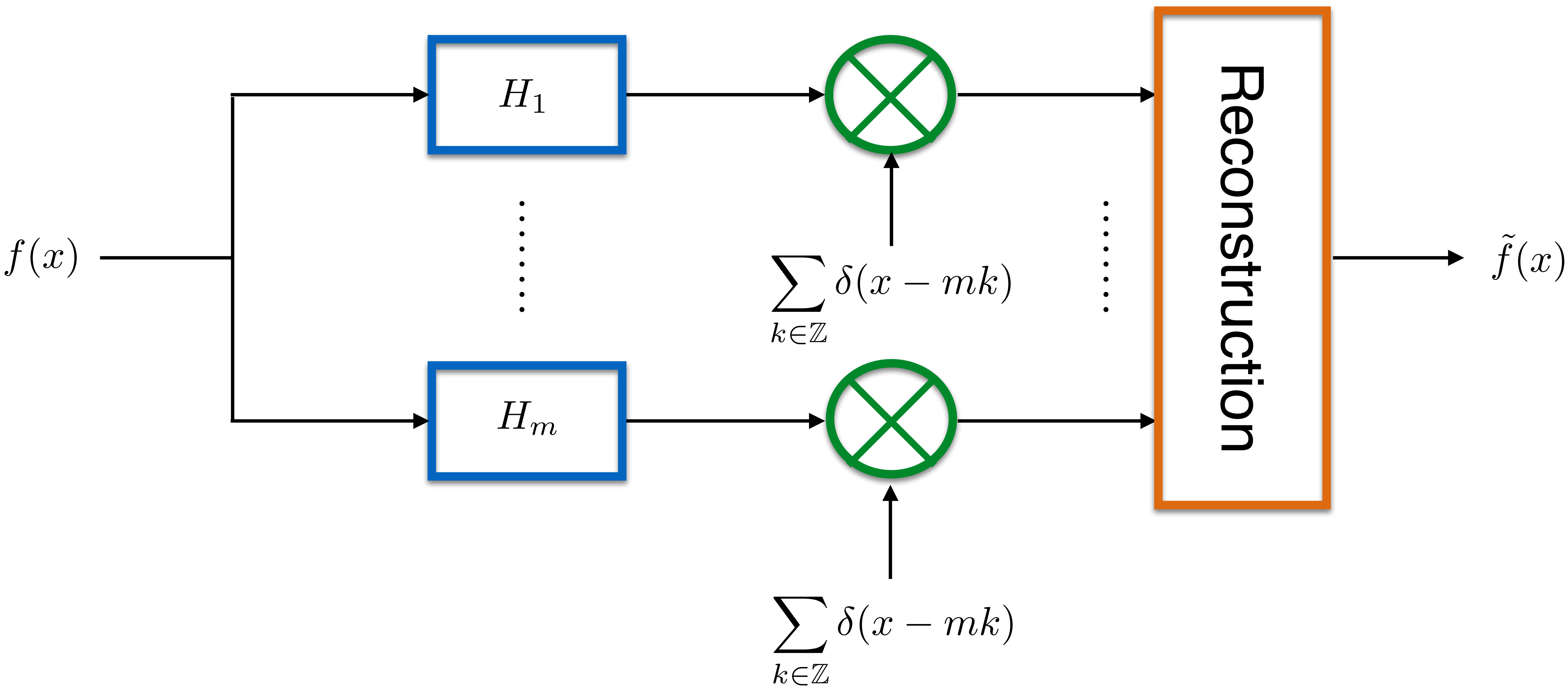
GENERALIZED SAMPLING

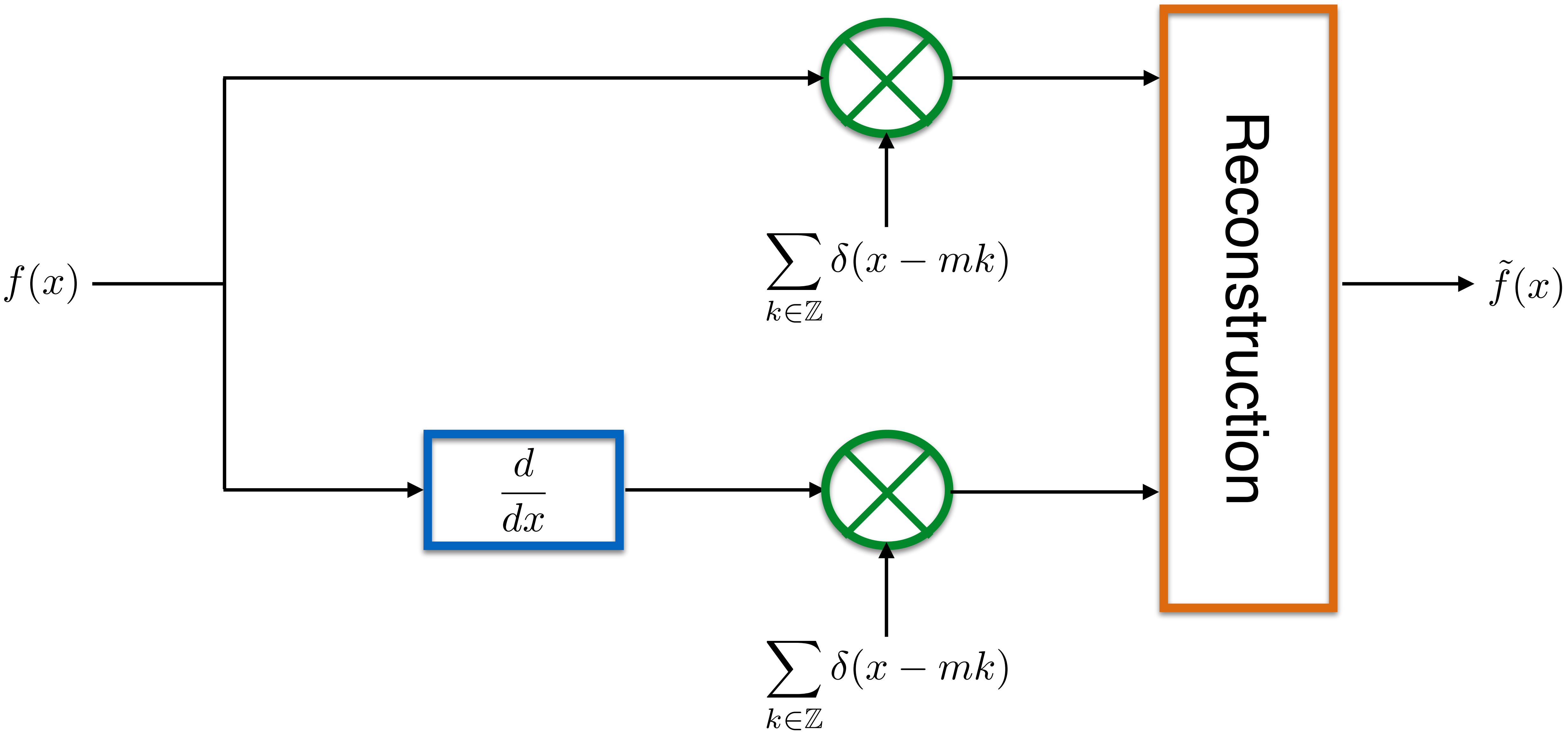
- Extension of **Shannon's** celebrated **sampling** theorem



- **Papoulis 1977:** Framework for **band-limited** functions
- **Unser & Zerubia 1998:** No **band-limited** constraint

DERIVATIVE SAMPLING

- **Special** case of generalized sampling
- Linear systems are **derivatives** of certain degrees
- In this work: Samples of the **function** and its **first-order** derivative



Finitely Generated Shift-Invariant Spaces

- (Integer) Shift-Invariant Subspaces [de Boor et al. 1994]

$$V \subseteq L_2(\mathbb{R}) : \quad f \in V \Rightarrow f(\cdot - k) \in V \quad \forall k \in \mathbb{Z}$$

- **Finitely-Generated** Shift-Invariant Spaces:

$$S(\phi) = \overline{\mathrm{Span}}(\{\phi(\cdot - k)\}_{k\in\mathbb{Z}}), \;\; S(\phi_1, \phi_2, \ldots, \phi_n) = S(\phi_1) + S(\phi_2) + \cdots + S(\phi_n)$$

- Assumption: Joint **Riesz** Condition

$$m\|\vec{a}\|_{\ell_2} \leq \left\| \sum_{n=1}^{N} \sum_{k\in\mathbb{Z}} a_n[k]\phi_n(\cdot - k) \right\|_{L_2} \leq M\|\vec{a}\|_{\ell_2}, \quad \vec{a}[k] = (a_1[k], a_2[k], \ldots, a_N[k])$$

- **Unique** and **Stable** Representation:

$$\forall f \in S(\phi_1, \phi_2, \ldots, \phi_n) : \quad f(\cdot) = \sum_{n=1}^{N} \sum_{k\in\mathbb{Z}} a_n[k]\phi_n(\cdot - k), \quad \forall n : a_n[\cdot] \in \ell_2(\mathbb{Z})$$

Derivative Sampling In Shift-Invariant Spaces

- We consider shift-invariant spaces with **two generators**.
    - **one-to-one** correspondence between the **coefficients** and **samples**

$$V = S(\phi_1, \phi_2) \qquad\qquad \phi_1, \phi_2 \in C^1(\mathbb{R})$$

- The target function $f \in W_2^1(\mathbb{R})$ is in a Sobolev space

- $\{f(k)\}_{k\in\mathbb{Z}}$ and $\{f'(k)\}_{k\in\mathbb{Z}}$ is given.

- Find $\tilde{f} \in V : \tilde{f}(k) = f(k), \quad \tilde{f}'(k) = f'(k), \quad \forall k \in \mathbb{Z}$

Finding The Coefficients

Fundamental Equations

$$\begin{cases} f(\ell) = \sum_{k\in\mathbb{Z}} a_1[k]\phi_1(\ell-k) + \sum_{k\in\mathbb{Z}} a_2[k]\phi_2(\ell-k), \\ f'(\ell) = \sum_{k\in\mathbb{Z}} a_1[k]\phi_1'(\ell-k) + \sum_{k\in\mathbb{Z}} a_2[k]\phi_2'(\ell-k). \end{cases}$$

Change of Notations

$$\begin{cases} \vec{f}[k] = (f(k), f'(k)), \vec{\phi}[k] = (\phi_i[k], \phi_i'[k]) \\ \vec{f}[\ell] = a_1 * \vec{\phi}_1[\ell] + a_2 * \vec{\phi}_2[\ell], \quad \forall \ell \in \mathbb{Z}, \end{cases}$$

Z-domain

$$\begin{cases} \vec{F}(z) = A_1(z)\vec{\Phi}_1(z) + A_2(z)\vec{\Phi}_2(z). \\ \mathbf{\Phi}(z) = \left[\vec{\Phi}_1(z) \quad \vec{\Phi}_2(z)\right] \end{cases}$$

Digital Filtering

$\mathbf{H} = [h_{i,j}]$ digital filter corresponds to $\mathbf{\Phi}^{-1}(z)$

$$\boxed{a_i[k] = h_{i,1} * f[k] + h_{i,2} * f'[k]}$$

Interpolatory Generators

- **Interpolatory Generators:** $\vec{a}[k] = \vec{f}[k], \quad \forall k \in \mathbb{Z}$

- Equivalently, when the digital filter is identity!

- How to construct interpolatory generators: $\vec{\phi}_{\text{int}}(x) = \sum_{k \in \mathbb{Z}} \mathbf{H}^T[k] \vec{\phi}(x-k)$

  It is **not** necessarily **compactly supported**

PROPERTIES OF RECONSTRUCTION METHOD

- **Computational Complexity** $\Leftrightarrow$ **Support size** of the generators
- **Approximation Power:** **Rate of convergence** to the target function when the **grid size** goes to zero

  In tie with the **polynomial reproducibilty** [Strang&Fix 1971]

  Polynomial reproducibility of degree up to M :

$$x^m = \sum_{n=1}^{N} \sum_{k \in \mathbb{Z}} p_n^{(m)}[k] \phi_n(x-k), \quad \forall x \in \mathbb{R} \backslash \{0\} \qquad m = 0, 1, \ldots, M$$

- Our main result: **Connecting** the two properties

Main Result: Tight Bound for Support Size

- What is the relation between **approximation power** and **complexity**?

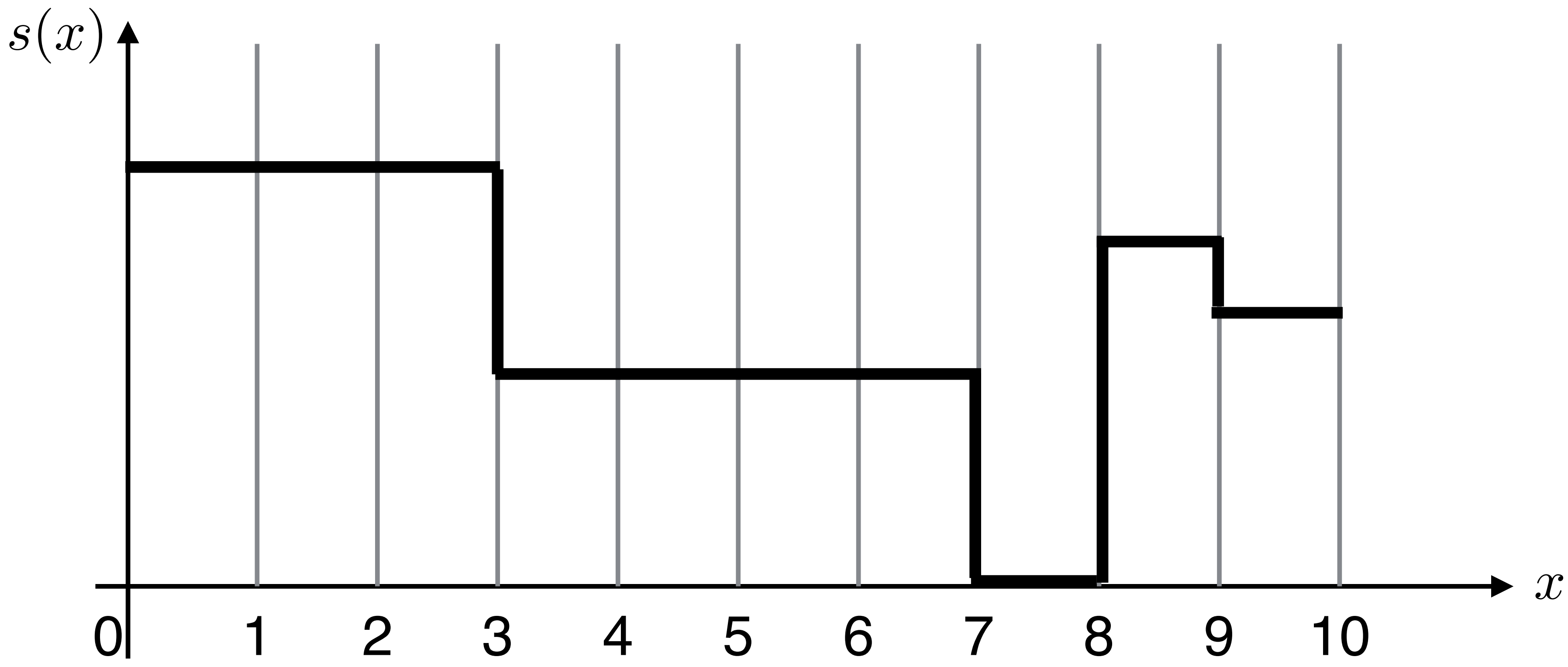
  Single Generated Subspaces: Answered by **Schoenberg in 1973**

- What about the case of **two generators** (derivative sampling)?

- Theorem: Let $V = S(\phi_1, \phi_2) \subseteq L_2(\mathbb{R})$ be a shift-invariant subspace with the ability to reproduce polynomials of degree up to M. Then
$$|\mathrm{Supp}(\phi_1)| + |\mathrm{Supp}(\phi_2)| \geq M + 1$$

PRACTICAL EXAMPLES: SPLINES

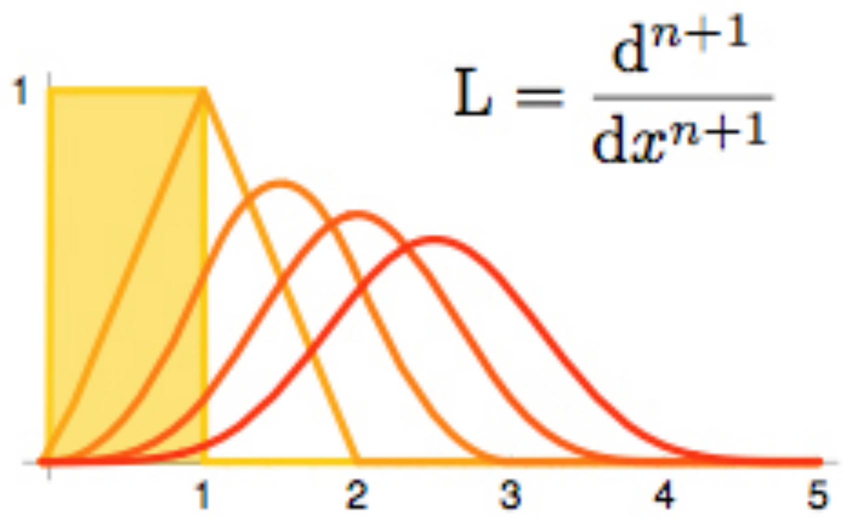
- **Splines** of degree n: piece-wise degree n polynomials that have $n-1$ continuous derivatives
- **Cardinal** splines: knots are located on the integer grid



Example: Cardinal Spline of Degree 0

B-SPLINES

- S_n: The set of cardinal splines of degree n
- V_n: The set of polynomials of degree n
- $V_n \subseteq S_n$
- β^n: **B-spline** of degree n
- **Smallest** support generator of S_n: $\mathrm{Supp}(\beta^n) = [0, n+1]$



Source: Splines: A Unifying Framework for Image Processing. M. Unser

Derivative Sampling and Splines

- The proposed space: $S_{n-1} + S_n$ for $n \geq 3$ (continuous derivative)

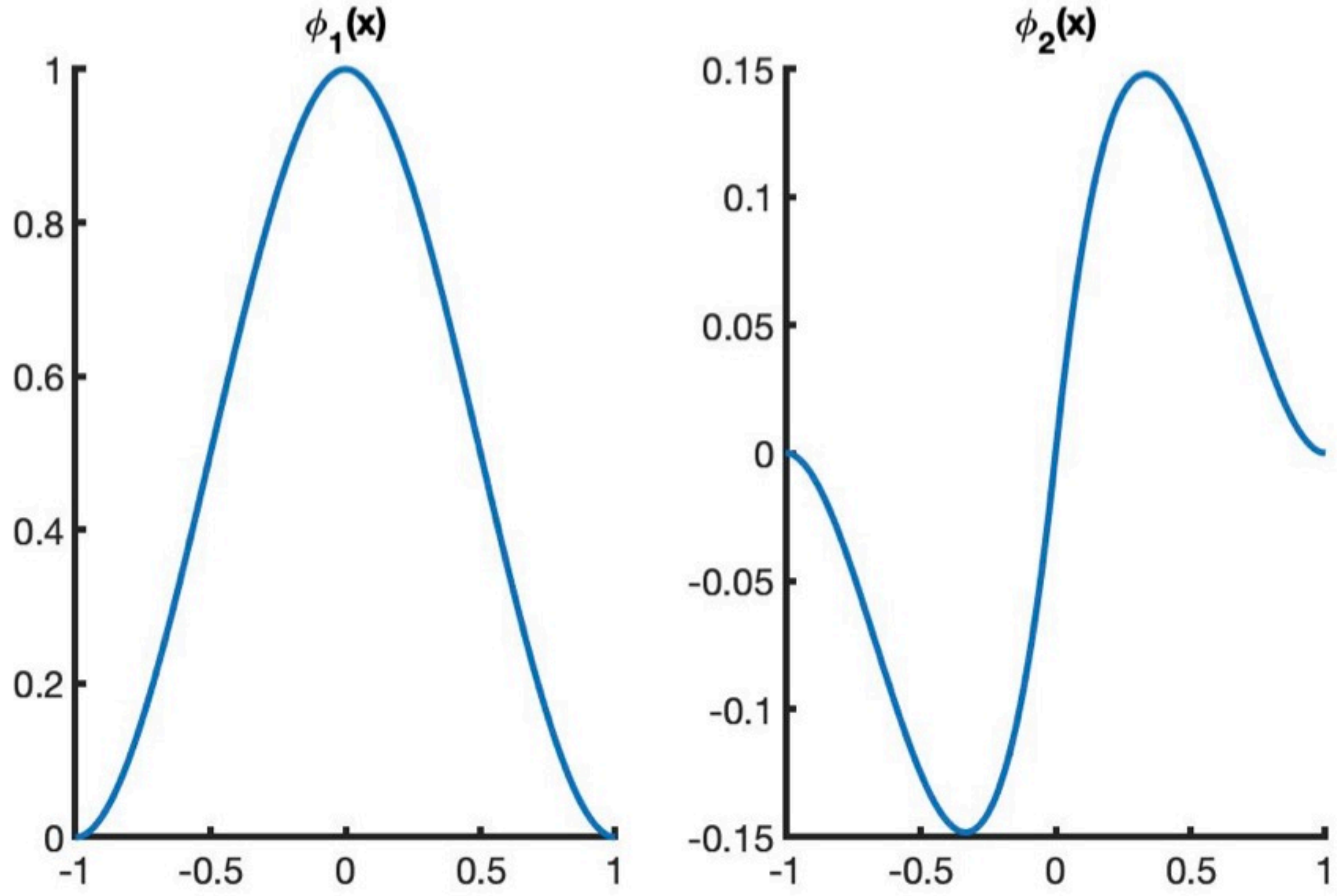
- **Canonical** generators: β^{n-1} and β^n

$\text{sum of support} = n + (n+1) = 2n + 1$ $\quad\quad$ $\text{optimal bound} = n + 1$

EXAMPLE: HERMITE SPLINES

- Lipow & Schoenberg 1973
- **Interpolatory**, **maximally localized**

Example of applications: curve parametrization [Uhlmann et al. 2016]

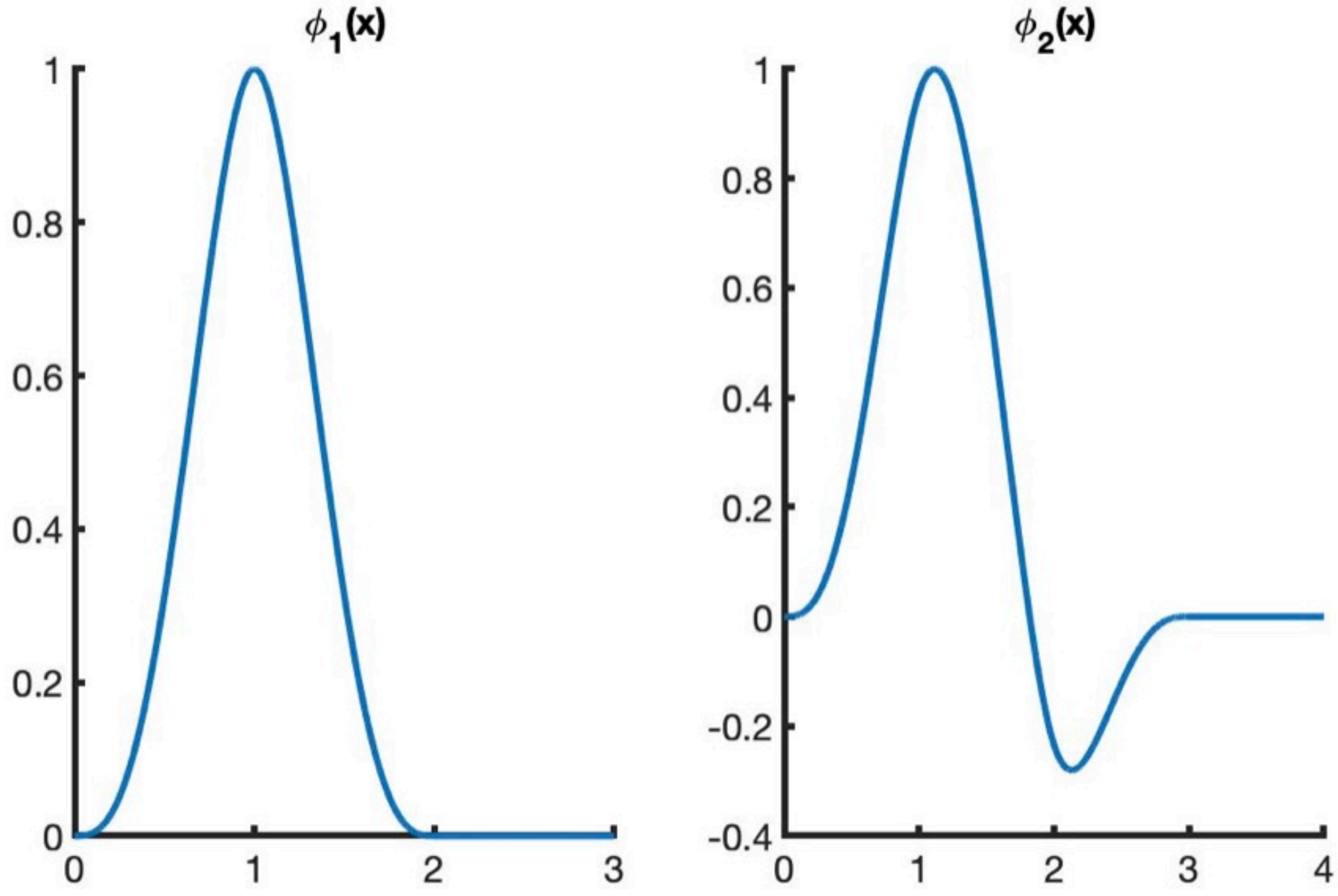


$S_2 + S_3$

How to Design?

- Toy Example: $S_3 + S_4$
- Sum of support of **optimal** generators = **5**
- Space of piece-wise polynomials of degree **4** in $C^2(\mathbb{R})$
- There is no function $f \in S_3 + S_4$ with $\mathrm{Supp}(f) = [0, 1]$
- The only possible choice for the size of optimal generators = **(2,3)**

Example of Optimal Generators



$$S_3 + S_4$$

CONCLUSION

- Consider derivative sampling over shift-invariant spaces.
- Describe the reconstruction method with recursive filtering.
- Provide a lower-bound on the sum of the support of the generators.
- It relies on the polynomial reproducibility of the search space.
- Construct optimal spline generators for derivative sampling.

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THANKS FOR YOUR ATTENTION!